

OPTIMAL REGULARITY FOR DEGENERATE OBSTACLE PROBLEMS

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ABSTRACT. In this paper we discuss the p -obstacle problem for both elliptic and parabolic equations. We prove several optimal results concerning the regularity of the solution. We are in particular interested in the point-wise regularity of the solution at free boundary points.

The most surprising result we prove is the one for the p -obstacle problem: Find the “smallest” u such that

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \leq 0, \quad u \geq \phi, \quad \text{in } B_1,$$

with $\phi \in C^{1,1}(B_1)$ and given boundary data on ∂B_1 . We prove that the solution is uniformly $C^{1,1}$ at free boundary points. The similar parabolic case gives us $C^{1,\frac{1}{p-1}}$ in the spatial variables and Lipschitz regularity in t , at free boundary points. This is obtained under the stronger assumptions that the boundary data is semi-monotone in t and the obstacle is C^2 .

1. INTRODUCTION

1.1. Problem formulation. In this paper we consider the the optimal regularity of minimizers of the the constrained p -Dirichlet energy

$$\int_{B_1} |\nabla v|^p dx, \quad v \in \mathbb{K} := \{w \in W^{1,p}(B_1) : w \geq \phi, w = g \text{ on } \partial B_1\},$$

where $B_1 \subset \mathbb{R}^n$ ($n \geq 2$) is the unit ball, and ϕ and g are given functions (in appropriate space). This is equivalent to finding the smallest function u such that

$$\Delta_p u \leq 0, \quad u \geq \phi,$$

given the boundary conditions on ∂B_1 . Here, and in the sequel, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplace operator.

Of particular interest is the set $\Omega = \{u > \phi\} \cap B_1$ and the free boundary $\Gamma = \partial\{u > \phi\} \cap B_1$. To better understand the free boundary, Γ , it is important to first understand the point-wise regularity of the solution u . We prove, the rather “unexpected” result, that the point-wise regularity of u at a free boundary point is the same as the regularity of the obstacle ϕ , at least up to $C^{1,1}$. That is if $\phi \in C^{1,1}$ then u leaves ϕ in a quadratic fashion. This surprising result implies, in turn, that the presence of the obstacle actually improves the regularity of the solution to the solution to the p -harmonic obstacle problem, at free boundary points.

In the second part of the paper we consider the p -parabolic obstacle problem. The p -parabolic obstacle problem amounts to finding the “smallest” function u , defined on $B_1 \times (0, T)$, with given boundary data, such that

$$\begin{cases} \Delta_p u - \frac{\partial u}{\partial t} \leq 0, \\ u \geq \phi. \end{cases}$$

For this, we prove the optimal growth of order $\frac{p}{p-1}$ in the spatial variables and of linear order in time under the assumption $\frac{\partial u}{\partial t} \geq -L$. This assumption is always satisfied if the same is true for the boundary data and the obstacle.

1.2. Known results. The non-degenerate elliptic obstacle problem, $p = 2$, is very well studied and the regularity properties of the solution are well known. It was proved by Frehse in [13] and Kinderlehrer in [15] (in two dimensions) that u is $C^{1,1}$, provided the same is true for the obstacle. Later in [6] it was proved that the free boundary, except at cusp-like points, is a C^∞ hypersurface. This result was sharpened even further in two dimensions by Monneau in [25]. A related but somewhat different problem was studied in [14] and [18]. See also [26] and [21] for regularity results relating to the p -harmonic obstacle problem.

For the parabolic obstacle problem, there is a series of papers [2], [5], [4] and [3], where optimal regularity as well as the free boundary regularity is proved in the non-degenerate and non-singular case when $p = 2$, for variable coefficients and right-hand side. In the papers [19] and [20], the right-hand side is allowed to be merely in L^p . In [27] the elliptic part of the operator is allowed to be fully nonlinear. A slightly more general free boundary problem of parabolic type is studied in [8], [12], [11] and [1].

In the p -parabolic case we refer the reader to the literature: [9], [16], [22], [23] and [17]. One of the authors studied a quite similar problem in [28].

1.3. Main idea. Roughly speaking, the main idea for the elliptic problem is the following: When the gradient is large then the equation is non-degenerate and classical $C^{1,1}$ -estimates apply. But on the other hand, when the gradient is small we can rescale quadratically and obtain uniform bounds using the weak Harnack inequality (which applies to supersolutions).

In the parabolic setting we make use of the ideas in [28]. For this we need a bound on u_t from below. This is what forces us to assume semi-monotonicity in t for the lateral boundary data (see Lemma 8).

Remark 1. *It is noteworthy, that our results, even though stated in terms of the obstacle problem, are of more general nature. Indeed, what we prove is that reasonably smooth super-solutions to the p -harmonic operator, must be as regular as a possible touching graphs from the below.*

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2. THE ELLIPTIC PROBLEM

In this part we treat the elliptic problem. Given an open, bounded set Ω and some boundary data given by the restriction of $g \in W^{1,p}(\Omega)$ to $\partial\Omega$, we say that u is a solution of the p -obstacle problem in Ω with obstacle $\phi \in C^{1,\beta}$, $g \geq \phi$, if u minimizes

$$\int_{\Omega} |\nabla u|^p dx$$

subject to $u \geq \phi$ in Ω and $u = g$ on $\partial\Omega$.

The first main result of this paper is the optimal growth at free boundary points.

Theorem 2. *Let $p \in (1, \infty)$, $\beta \in (0, 1]$ and u be a solution to the p -obstacle problem in B_1 with obstacle $\phi \in C^{1,\beta}(B_1)$. Suppose further*

$$\|\phi\|_{C^{1,\beta}(B_1)} \leq N.$$

Then for any point $y \in \Gamma \cap B_{1/2}$ and for $r < 1/2$

$$(1) \quad \sup_{x \in B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq CNr^{1+\beta},$$

where $C = C(\beta, p)$.

Proof. By simply considering the normalized function $u/2N$, we can assume that u solves the p -obstacle problem with obstacle ϕ satisfying $\|\phi\|_{C^{1,\beta}(B_1)} \leq 1/2$. Then at any free boundary point y , we have $|\nabla u(y)| = |\nabla \phi(y)| \leq 1/2$. The proof is now divided into different cases. The correctly scaled estimate is then obtained in the end by multiplying the constant with $2N$.

Case 1: When $|\nabla u(y)| < r^\beta < (1/2)^\beta$: When $|\nabla u(y)| < r^\beta$ it follows from the triangle inequality that

$$\sup_{x \in B_r(y)} |u(x) - u(y)| \leq Cr^{1+\beta} \Rightarrow \sup_{x \in B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq (C + 1)r^{1+\beta}.$$

It is therefore enough to prove

$$\sup_{B_r(y)} (u - u(y)) \leq Cr^{1+\beta},$$

for some constant $C = C(\beta, p)$. To this end we define the rescaled functions

$$\tilde{\phi}(x) = \frac{\phi(rx + y) - \phi(y)}{r^{1+\beta}}$$

and

$$\tilde{u}(x) = \frac{u(rx + y) - u(y)}{r^{1+\beta}}.$$

We may estimate the L^∞ norm of $\tilde{\phi}$ according to

$$\|\tilde{\phi}\|_{L^\infty(B_1)} = \left| \frac{\phi(rx + y) - \phi(y)}{r^{1+\beta}} \right|_{L^\infty(B_1)} \leq$$

$$\leq \left\| \frac{\phi(rx + y) - \phi(y) - r \nabla \phi(y) \cdot (x - y)}{r^{1+\beta}} \right\|_{L^\infty(B_1)} + \left\| \frac{\nabla \phi(y) \cdot (x - y)}{r^\beta} \right\|_{L^\infty(B_1)} \leq 1,$$

where we used that $\nabla \phi = \nabla u$ at a free boundary point by Proposition 10 in the appendix. Since the minimization problem is invariant under dilations, \tilde{u} is a solution to the obstacle problem in B_1 with $\tilde{\phi}$ as obstacle. Thus \tilde{u} is p -superharmonic. The weak Harnack inequality for p -superharmonic functions, see for instance Theorem 3.13 in [24], applied to the non-negative function $\tilde{u} + 1 \geq -\|\tilde{\phi}\|_{L^\infty} + 1 \geq 0$ implies

$$\|\tilde{u} + 1\|_{L^q(B_{\frac{3}{4}})} \leq C_1(p) \inf_{B_{\frac{1}{2}}}(\tilde{u} + 1) \leq C_1(p) (\tilde{\phi}(0) + 1) \leq 4C_1(p) = C_2(p).$$

for some $q > 1$. Now, let

$$v = \max(\tilde{u} + 1, \sup_{B_1} \tilde{\phi} + 1).$$

Then $\Delta_p v \geq 0^1$ and thus from the sup-estimate for subsolutions (cf. Corollary 3.10 in [24]) we can conclude together with the estimate above that

$$\sup_{B_{\frac{1}{2}}} v \leq C_3(p) \|v\|_{L^q(B_{\frac{3}{4}})} \leq C_3(p) C_2(p).$$

This implies, upon relabeling the constants, that

$$(2) \quad \sup_{B_{\frac{1}{2}}} \tilde{u} \leq C(p).$$

Since moreover $\tilde{u} \geq \tilde{\phi} \geq -1$, \tilde{u} is uniformly bounded in $L^\infty(B_{1/2})$, which implies the desired estimate, for $r < 1/4$. In order to obtain the estimate for $r \in (1/4, 1/2)$ one just needs to increase the constant by $2^{1+\beta}$.

Case 2: When $|\nabla u(y)| \geq r^\beta$, $r < 1/2$: From Case 1 we know that

$$(3) \quad \sup_{B_{ry}(y)} u(x) \leq C(p) r_y^{1+\beta}$$

where $r_y^\beta = |\nabla u(y)|$. Let

$$\tilde{\phi}(x) = \frac{\phi(r_y x + y) - \phi(y)}{r_y^{1+\beta}}.$$

Then $|\nabla \tilde{\phi}(0)| = 1$. Define also

$$\tilde{u}(x) = \frac{u(r_y x + y) - u(y)}{r_y^{1+\beta}}.$$

Then \tilde{u} solves the p -obstacle problem in B_1 with $\tilde{\phi}$ as an obstacle. Moreover, from the assumption $\|\phi\|_{C^{1,\beta}} \leq 1/2$,

$$\|\tilde{\phi}\|_{C^{1,\beta}(B_{1/2})} \leq C_4,$$

¹Observe that $v = \max(\tilde{u} - \sup \tilde{\phi}, 0) + 1 + \sup \tilde{\phi}$ and $\tilde{u} - \sup \tilde{\phi}$ is p -harmonic in the set $\{\tilde{u} - \sup \tilde{\phi} > 0\}$ and it is zero outside this set. By taking a test function of the form $\phi \eta((\tilde{u} - \sup \tilde{\phi})^+)$, with $\phi \in C_0^\infty(B_1)$ and η a linear approximation of the identity, it follows that $\Delta_p v \leq 0$.

and by (3) \tilde{u} is uniformly bounded in $L^\infty(B_{1/2})$. From Proposition 10 it follows that

$$\|\tilde{u}\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \leq C_5, \quad \alpha = \alpha(p), \quad C_5 = C_5(p).$$

Consequently, there is $r_0 = r_0(p)$ so that $|\nabla \tilde{u}| \geq 1/2$ in B_{r_0} . Hence, \tilde{u} is a uniformly bounded solution to the obstacle problem with C^α -coefficients in B_{r_0} for a uniformly elliptic operator with $C^{1,\beta}$ obstacle. From Proposition 10 and Proposition 11

$$\|\tilde{u}\|_{C^{1,\beta}(B_{r_0})} \leq C(p, \beta).$$

Scaling back we obtain

$$\|u\|_{C^{1,\beta}(B_{r_0 r_y}(y))} \leq C(p, \beta),$$

which in particular implies

$$\sup_{B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq C r^{1+\beta},$$

for $r < r_0 r_y = r_0 |\nabla u(y)|^{\frac{1}{\beta}}$.

Conclusion: In both Case 1 and Case 2 we concluded that

$$\sup_{B_r(y)} |u(x) - u(y) - x \cdot \nabla u(y)| \leq C r^{1+\beta},$$

for all $r < 1/2$ such that $r \leq r_0 |\nabla u(y)|^{\frac{1}{\beta}}$ (Case 2) and when $r > |\nabla u(y)|^{\frac{1}{\beta}}$ (Case 1). Therefore we only need to fill the gap when

$$(4) \quad r_0 |\nabla u(y)|^{\frac{1}{\beta}} < r < |\nabla u(y)|^{\frac{1}{\beta}}.$$

Assume that r is in the interval specified in (4). Then

$$\sup_{B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \leq \sup_{B_{r_y}(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)|.$$

Hence,

$$\begin{aligned} \sup_{B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| &\leq \sup_{B_{r_y}(y)} |u(x) - u(y) - (x - y) \cdot \nabla u(y)| \\ &\leq C N r_y^{1+\beta} = C \frac{r_y^{1+\beta}}{r^{1+\beta}} r^{1+\beta} \leq \frac{C}{r_0^{1+\beta}} r^{1+\beta}, \end{aligned}$$

we thus have the estimate for all $r < 1/2$. To obtain the estimate for the original u (not rescaled by a factor $2N$) one just needs to multiply the constant C with $2N$. \square

3. NON-DEGENERACY AND POROSITY OF THE FREE BOUNDARY

In this section we prove by standard arguments that the difference $u - \phi$ cannot decay faster than quadratic around free boundary points. This combined with the optimal quadratic growth implies, by a standard argument, that the free boundary Γ is porous. We recall that $\Gamma \cap B_{1/2}$ is said to be *porous* if there exists a $\delta > 0$ such that for every $y \in \Gamma \cap B_{1/2}$ and $r \in (0, 1/4)$

$$\frac{|\Gamma \cap B_r(y)|}{|B_r(y)|} \leq 1 - \delta.$$

Since this directly implies that Γ has no Lebesgue points it follows that the free boundary has measure zero. The notion of porosity was introduced in [10]; See also the survey [29].

Proposition 3. *Let $p \in (2, \infty)$ and let u be a solution to the p -obstacle problem in B_1 with obstacle $\phi \in C^2(B_1)$. Suppose further that $\Delta_p \phi < 0$. Then there is a constant $\varepsilon = \varepsilon(\sup \Delta_p \phi)$ such that for any $x^0 \in \Gamma$ and $r < \text{dist}(x^0, \partial B_1)$ there holds*

$$\sup_{\partial B_r(x^0) \cap \{u > \phi\}} (u - \phi) \geq \varepsilon r^2.$$

Proof. The proof is standard. Take $y \in \{u > \phi\}$. Let $v(x) = \phi(x) + \varepsilon|x - y|^2$, where ε is chosen small enough such that $\Delta_p v < 0$. This is indeed possible since $\Delta_p v$ is continuous with respect to ε . Pick $r < \text{dist}(x^0, \partial B_1)$. Then $\Delta_p u = 0 \geq \Delta_p v$ in $\{u > \phi\} \cap B_r(x^0)$. Moreover, $u(y) \geq \phi(y) = v(y)$. From the comparison principle it follows that there is $z_y \in \partial(\{u > \phi\} \cap B_r(x^0))$ such that $u(z_y) \geq v(z_y)$. Since $u < v$ on $B_r(x^0) \cap \partial\{u > \phi\}$ there must be $z_y \in \{u > \phi\} \cap \partial B_r(x^0)$ such that $u(z_y) \geq v(z_y)$. The result follows by continuity and by letting $y \rightarrow x^0$. \square

In order to prove that the free boundary is porous we also need the following gradient estimate.

Lemma 4. *Let $p \in (2, \infty)$ and let u be a solution to the p -obstacle problem in B_1 with obstacle $\phi \in C^2(B_1)$ and $\|\phi\|_{C^{1,1}(B_1)} \leq N$. Suppose $x^0 \in \Gamma$. Then there is $C = C(p)$, (with N as in Theorem 2), such that*

$$\sup_{B_r(x^0)} |\nabla u - \nabla \phi| \leq CNr.$$

for $r < 1/2$.

Proof. The proof is similar to the proof of Theorem 2; We assume $N = 1$ and split the proof into two cases.

Case 1: When $|\nabla u(x^0)| \leq r$. As in Case 1 of the proof of Theorem 2, with $\beta = 1$, we can rescale u and obtain (2). Again, \tilde{u} is a uniformly bounded solution to the p -obstacle problem in B_1 with $\tilde{\phi}$ as obstacle. In particular, $\|\tilde{\phi}\|_{C^{1,1}} \leq 1/2$. Then

$$\sup_{B_{\frac{1}{4}}} |\nabla \tilde{u}| \leq C(p),$$

by Proposition 10. Rescaling back this implies

$$\sup_{B_{\frac{r}{4}}} |\nabla u| \leq C(p)r.$$

Case 2: $|\nabla u(x^0)| = r_{x_0} > r$. We proceed and define \tilde{u} and $\tilde{\phi}$ as in Step 2 in the proof of Theorem 2. As before, \tilde{u} solves the obstacle problem for a uniformly elliptic operator with C^α -coefficients B_{r_0} where $r_0 = r_0(p)$. If we denote by L the operator

$$L = \operatorname{div}(|\nabla \tilde{u}|^{p-2} \nabla \cdot),$$

then $\tilde{u} - \tilde{\phi} \geq 0$ and

$$L(\tilde{u} - \tilde{\phi}) = (-L\tilde{\phi})\chi_{\{\tilde{u} > \tilde{\phi}\}} \text{ in } B_{r_0}.$$

Since $\|L\tilde{\phi}\|_{L^\infty(B_{r_0})} \leq C$, Calderón-Zygmund estimates imply

$$\sup_{B_{\frac{\rho}{2}}} |\nabla \tilde{u} - \nabla \tilde{\phi}| \leq C \left(\frac{1}{\rho} \sup_{B_\rho} |\tilde{u} - \tilde{\phi}| + \rho \right), \quad C = C(p)$$

for $\rho \leq r_{x_0}$. Scaling back,

$$\sup_{B_{\frac{s}{2}}} |\nabla u - \nabla \phi| \leq C \left(\frac{1}{s} \sup_{B_s} |u - \phi| + s \right) \leq Cs,$$

for $s \leq r_0 r_{x_0}$, upon relabelling the constant C . Here we used Theorem 2 to bound $u - \phi$.

Combining the two steps above, we obtain the estimate for $r \leq r_0 r_{x_0}$ and for $r_{x_0} \leq r < 1/2$. By increasing the constant we obtain the same estimate for any $r < 1/2$. Finally, by multiplying with $2N$ we obtain the desired estimate in terms of N . \square

Corollary 5. *Under the assumptions in Proposition 3, the free boundary is porous. In particular it has Lebesgue measure zero.*

Proof. Take $x^0 \in \Gamma$. By Proposition 3, for r small enough, there is $y \in \partial B_r(x^0)$ such that

$$u(y) - \phi(y) \geq \varepsilon r^2.$$

Now take $\rho < 1$ so that $B_{\rho r}(y) \subset B_{2r}(x^0)$. Lemma 4 implies that $|\nabla(u - \phi)| \leq Cr$ in $B_{\rho r}(y)$. Thus, for $z \in B_{\rho r}(y)$

$$u(z) - \phi(z) \geq u(y) - \phi(y) - |z - y| \sup_{B_{\rho r}(y)} |\nabla u| \geq \varepsilon r^2 - C' \rho r^2 = r^2(\varepsilon - C\rho) > 0$$

whenever ρ is small enough. Hence, $B_{\rho r}(y) \subset \{u > \phi\}$ for ρ small enough, which means exactly that Γ is porous. From Lebesgue's density theorem it follows that Γ has zero Lebesgue density. \square

4. THE PARABOLIC PROBLEM

In this part we treat the parabolic problem introduced earlier. With $q = \frac{p}{p-1}$, we introduce the notations

$$Q_r^-(x, t) = B_r(x, t) \times (-r^q + t, t], \quad \partial_p Q_r^-(x, t) = \partial B_r(x, t) \times (-r^q + t, t] \cup B_r(x, t) \times \{t\},$$

with the simplification $Q_r^- = Q_r^-(0, 0)$. Given boundary data g on $\partial_p Q_r^-$, we say that u is a solution of the p -parabolic obstacle problem in Q_r^- with obstacle ϕ if u satisfies

$$(5) \quad \begin{cases} \max(\Delta_p u - u_t, u - \phi) = 0 & \text{in } Q_r^-, \\ u = g & \text{on } \partial_p Q_r^-. \end{cases}$$

In this section we prove that if a solution satisfies $u_t \geq -L$ for some constant $L \geq 0$ then it has the optimal growth of order $\frac{p}{p-1}$ in x of order one in time, at free boundary points. We also give an example of assumptions under which the solution satisfies $u_t \geq -L$. Our approach is essentially equivalent to the one in [28].

The main result of this section is stated below:

Theorem 6. *Let $p \in (2, \infty)$ and u be a solution to the p -parabolic obstacle problem in Q_1^- with obstacle $\phi \in C^2(Q_1^-)$. Suppose further that*

$$u_t \geq -L, \quad \|u\|_{L^\infty(Q_1^-)} \leq M, \quad \|\phi\|_{C^2(Q_1^-)} \leq N.$$

Then for any point $(y, s) \in \Gamma \cap Q_{1/2}^-$ and for $r < 1/4$

$$(6) \quad \sup_{(x,t) \in Q_r^-(y,s)} |u(x, t) - u(y, s) - (x - y) \cdot \nabla u(y, s)| \leq Cr^q, \quad q = \frac{p}{p-1}.$$

where $C = C(p, L, M, N)$.

As in the elliptic case (Theorem 2) the proof splits into two cases depending on whether ∇u is small or not at the free boundary point. However, there are a major differences between the behavior of the elliptic and the parabolic p -obstacle problem. One is that the elliptic operator is invariant under multiplication, which is not true for the parabolic operator. Another is that the p -parabolic operator does not have a strong maximum principle which forces us to use a slightly different approach. This also explains why the Theorem 6 is slightly weaker in that we do not get a linear dependence in N on the right hand side of (6). One should also note that the estimate obtained in Theorem 2 is independent of the L^∞ -norm the solution, while the estimate obtained in Theorem 6, does depend on the L^∞ -norm of the solution. We believe that this is rather an artifact of the method of proof than a structural difference between the two problems.

The next proposition gives the estimate corresponding to Case 1 in the proof of Theorem 2. Once we have proved this proposition the proof of Theorem 6 will follow easily.

Proposition 7. *Assume the hypotheses of Theorem 6 and that $(0, 0) \in \Gamma$. If $|\nabla u(0, 0)| \leq r^{q-1}$ and $r < 1/2$ then*

$$S_r := \sup_{Q_r^-} |u(x, t) - u(0, 0)| \leq C_1 r^q, \quad q = \frac{p}{p-1}, \quad C_1 = C_1(p, L, M, N).$$

Proof. We assume that $|\nabla u(0, 0)| \leq r^{q-1}$. If there exist a C such that

$$(7) \quad S_r \leq C r^q$$

then we are done. Notice that if $r \geq 1/2$ then (7) is obviously true for some $C(M)$, since $\|u\|_{L^\infty} \leq M$. We will show that there exists a C such that

$$(8) \quad S_r \leq \max \left(\max_{2^k r \leq 1, k \geq 0} 2^{-kq} S_{2^k r}, C r^q \right),$$

for $r \leq 1/2$. Notice that if (8) is true for every r and $S_r \leq C$ for $1 > r \geq \frac{1}{2}$ then the proposition follows. Suppose that (8) is false. Then there are sequences r_j , u_j and obstacles ϕ_j such that $\|u_j\|_{L^\infty} \leq M$, $\|\phi_j\|_{C^2} \leq N$, $|\nabla u_j(0, 0)| \leq r_j^{q-1}$ and

$$S_{r_j} \geq \max \left(\max_{2^k r_j \leq 1, k \geq 0} 2^{-kq} S_{2^k r_j}, j r_j^q \right).$$

We will assume that r_j is chosen so that

$$(9) \quad S_r \leq \max \left(\max_{2^k r \leq 1, k \geq 0} 2^{-kq} S_{2^k r}, j r^q \right)$$

for all $r > r_j$. This is clearly possible since S_r is continuous in r and $S_{1/2}$ is bounded. Define the functions

$$v_j(x, t) = \frac{u_j(r_j x, r_j^p S_{r_j}^{2-p} t) - u_j(0, 0)}{S_{r_j}}.$$

Then v_j solves the p -parabolic obstacle problem in Q_1^- with the obstacle

$$\psi_j(x, t) = \frac{\phi_j(r_j x, r_j^p S_{r_j}^{2-p} t) - u_j(0, 0)}{S_{r_j}} \rightarrow 0,$$

where we have used that $\|\phi_j\|_{C^2} \leq N$ and that $u_j(0, 0) = \phi_j(0, 0)$ since the origin is a free boundary point. Moreover,

- (1) $v_j(0, 0) = 0$ and $|\nabla v_j(0, 0)| \leq \frac{1}{j} \rightarrow 0$;
- (2) by (9)

$$\sup_{Q_{2^k}^-} |v_j| \leq \sup_{Q_{2^k r_j}^-} \frac{|u_j|}{S_{r_j}} \leq 2^{qk},$$

for all k such that $2^k r_j \leq 1$.

We may use this to calculate

$$\begin{aligned} \sup_{B_1 \times (-1, 0]} |v_j| &= \sup_{B_r \times (-S_{r_j}^{2-p} r_j^p, 0)} \frac{u_j(x, t)}{S_{r_j}} \\ &\geq \sup_{B_r \times (-r_j^q, 0)} \frac{u_j(x, t)}{S_{r_j}} - L \frac{|r_j^q - S_{r_j}^{2-p} r_j^p|}{S_{r_j}} \geq 1 - \frac{L}{j}, \end{aligned}$$

where the last estimate is due to the assumption $u_t \geq -L$.

Before we can get our desired contradiction we need to control the behavior in time of v_j . From Corollary 13 we can conclude

$$\begin{aligned} |v_j(x, t) - v_j(x, s)| &\leq \frac{|u_j(r_j x, S_{r_j}^{2-p} r_j^p s) - u_j(r_j x, S_{r_j}^{2-p} r_j^p t)|}{S_{r_j}} \\ &\leq C \frac{\sup_{Q_{r_j}} |u_j|}{S_{r_j} r_j^{q\alpha}} |S_{r_j}^{2-p} r_j^p|^\alpha |t - s|^\alpha \\ &\leq C |S_{r_j}^{2-p} r_j^{p-q}|^\alpha |t - s|^\alpha \leq C j^{2-p} |t - s|^\alpha \rightarrow 0, \end{aligned}$$

since $p > 2$.

Hence, since the v_j s are locally bounded and solve the p -parabolic obstacle problem with uniformly smooth obstacle (in fact going to zero), we can extract a subsequence converging locally uniformly in $\mathbb{R}^n \times \mathbb{R}^-$ to a limit function $v_0 = v_0(x)$ which is independent of t satisfying

- (i) $v_0(0, 0) = |\nabla v_0(0, 0)| = 0$;
- (ii) $\sup_{B_1 \times (-1, 0]} |v_j| \geq 1$;
- (iii) $\Delta_p v_0 - (v_0)_t \leq 0$;

Since v_0 does not depend on t , also $\Delta_p v_0 \leq 0$. Moreover, (ii) implies $\sup_{B_1} v_0 \geq 1$. But then the fact that $v_0(0) = 0$ implies $v_0 \equiv 0$, via the strong minimum principle for the p -Laplacian. This is a contradiction. \square

Proof of Theorem 6. We may, by means of a translation of the problem, assume that $(y, s) = (0, 0) \in \Gamma$.

Notice that if $r \geq |\nabla u(0, 0)|^{\frac{1}{p-1}}$ then we may apply Proposition 7 directly implies that

$$(10) \quad \sup_{Q_r^-(y, s)} |u| \leq C_1 r^q, \quad C_1 = C_1(p, L, M, N)$$

It is therefore enough to show the Theorem for $r < |\nabla u(0, 0)|^{\frac{1}{p-1}}$.

Assume $r < |\nabla u(0, 0)|^{\frac{1}{p-1}}$ and $r < 1/2$. Define $\rho = |\nabla u(0, 0)|^{\frac{1}{q-1}}$ and the rescaled function

$$v(x, t) = \frac{u(\rho x, \rho^q t)}{\rho^q}.$$

By (10)

$$\sup_{Q_1^-} |v| \leq C_1.$$

Moreover, v solves the p -parabolic obstacle problem in Q_1^- with

$$\tilde{\phi}(x, t) = \frac{\phi(\rho x, \rho^q t)}{\rho^q},$$

as obstacle, so that $\|\tilde{\phi}\|_{C^2(Q_{1/2}^-)} \leq \tilde{N} = \tilde{N}(N)$. Hence, from Proposition 12, there are constants γ and A , depending on p, L, M and N , so that

$$\|u\|_{C_x^{1,\gamma}(Q_{\frac{1}{4}}^-)} \leq A.$$

We also observe that $|\nabla v(0, 0)| = 1$. Hence, we can find a small $\tau = \tau(p, L, M, N)$ so that $|\nabla v| > 1/2$ in Q_τ^- . Then v is still uniformly bounded in Q_τ^- and solves the obstacle problem in Q_τ^- for a uniformly elliptic operator with uniformly Hölder continuous coefficients. From Proposition 14 we conclude that

$$\sup_{Q_r^-(y, s)} |v(x, t) - v(y, s) - (x - y) \cdot \nabla v(y, s)| \leq C_2 r^q, \quad C_2 = C_2(p, L, M, N)$$

for $r < \tau$. Scaling back to the original function u we obtain

$$\sup_{Q_s^-(y, s)} |v(x, t) - v(y, s) - (x - y) \cdot \nabla v(y, s)| \leq C_2 s^q,$$

for $s < \tau\rho$. Hence, recalling (10), we have the desired estimate for $0 < r < \tau\rho$ and $r > \rho$, with constant $\max(C_1, C_2)$. The exact same arguments as in the conclusion of the proof of Theorem 3 now imply that the theorem upon increasing the constant with a factor ρ^{-q} . \square

The assumption that $u_t \geq -L$ in Theorem 6 is rather unsatisfactory since we do not know if it is true in general even for solutions of the equation $u_t = \Delta_p u$, without the presence of an obstacle. However, if the obstacle and the boundary data on the lateral boundary have their time derivatives bounded from below, so does the solution, as is shown below. In the following lemma we will, for notational convenience, assume that the solution is defined in Q_1^+ instead of Q_1^- . Here

$$Q_r^+(x, t) = B_r(x, t) \times (t, t + r^q], \quad Q_r^+ = Q_r^+(0, 0).$$

Lemma 8. *Let $p > 2$. Assume that u is a solution to the p -parabolic obstacle problem in Q_1^+ with obstacle ϕ and boundary data f and initial data g . Suppose further that $f_t \geq -N$ and $\phi_t \geq -N$ for some constant $N \geq 0$ in Q_1^+ . Then, for $t \geq \frac{1}{2}$,*

$$u_t \geq -L = -C(p)(N + \|\phi\|_{L^\infty}).$$

Proof. Let M be a large constant, to be determined later, and consider $\tilde{u} = u + M$, $\tilde{\phi} = \phi + M$, $\tilde{f} = f + M$ and $\tilde{g} = g + M$. Clearly \tilde{u} solves the p -parabolic obstacle problem with $\tilde{\phi}$ as obstacle and \tilde{f} and \tilde{g} as boundary data.

Now, let $v(x, t) = k^{\frac{1}{p-2}} \tilde{u}(x, kt)$ for some $k \in (1, 2)$. Then $v(x, t)$ solves a p -parabolic obstacle problem in $B_1(0) \times (0, k^{-1}) \subset Q_1^+$. Since $k^{\frac{1}{p-2}} > 1$ it automatically follows that $v(x, 0) > \tilde{u}(x, 0)$, given that $M \geq \|\phi\|_{L^\infty}$. With the aim of using the comparison principle and conclude that $v(x, t) \geq \tilde{u}(x, t)$, we claim that $v(x, t) \geq \tilde{u}(x, t)$ on $\partial B_1 \times (0, k^{-1})$ and that $k^{\frac{1}{p-2}} \tilde{\phi}(x, kt) \geq \tilde{\phi}(x, t)$ as well.

Indeed, let $h(x, t)$ be a function with distributional time derivative bounded from below by $-N$: $h(x, t + \lambda) \geq h(x, t) - N\lambda$ for $\lambda > 0$. Then

$$(11) \quad \begin{aligned} k^{\frac{1}{p-2}} h(x, kt) - h(x, t) &\geq k^{\frac{1}{p-2}} (h(x, t) - N(k-1)t) - h(x, t) \geq \\ &\geq \left(k^{\frac{1}{p-2}} - 1\right) h(x, t) - N(k-1)t. \end{aligned}$$

The expression in (11) is non-negative if

$$h(x, t) \geq \frac{N(k-1)t}{k^{\frac{1}{p-2}} - 1} \geq c(p)N.$$

In particular, if we choose $M > CN + \|\phi\|_{L^\infty}$ then it follows with $h(x, t) = \tilde{\phi}(x, t)$ (or $h = \tilde{f}$ since $\tilde{f} \geq \tilde{\phi}$) that

$$k^{\frac{1}{p-2}} \tilde{\phi}(x, kt) \geq \tilde{\phi}(x, t) \quad \left(k^{\frac{1}{p-2}} \tilde{f}(x, kt) \geq \tilde{f}(x, t)\right).$$

Therefore, for any $k \in (1, 2)$, v solves the p -parabolic obstacle problem in $B_1(0) \times (0, k^{-1})$ with obstacle $k^{\frac{1}{p-2}} \tilde{\phi}(x, kt) \geq \tilde{\phi}(x, t)$, boundary data $k^{\frac{1}{p-2}} \tilde{f}(x, kt) \geq \tilde{f}(x, t)$ and initial data $k^{\frac{1}{p-2}} \tilde{g}(x) \geq \tilde{g}(x)$. Hence, $v(x, t) \geq \tilde{u}(x, t)$ by the comparison principle.

To calculate the time derivative of $\tilde{u}(x, t)$ we calculate, with $k = 1 + \frac{h}{t}$

$$(12) \quad \begin{aligned} \liminf_{h \rightarrow 0} \frac{\tilde{u}(x, t+h) - \tilde{u}(x, t)}{h} &= \liminf_{h \rightarrow 0} \frac{k^{-\frac{1}{p-2}} v(x, t) - \tilde{u}(x, t)}{h} \geq \\ &\geq \liminf_{h \rightarrow 0} \frac{v(x, t) - \tilde{u}(x, t)}{h} + \liminf_{h \rightarrow 0} \frac{\left(k^{-\frac{1}{p-2}} - 1\right) v(x, t)}{h} \geq \\ &\geq \liminf_{h \rightarrow 0} \frac{\left(k^{-\frac{1}{p-2}} - 1\right) v(x, t)}{h}, \end{aligned}$$

since $v(x, t) \geq \tilde{u}(x, t)$. Also, by continuity of \tilde{u} we may calculate

$$(13) \quad \liminf_{h \rightarrow 0} \frac{\left(k^{-\frac{1}{p-2}} - 1\right) v(x, t)}{h} \geq -\frac{\tilde{u}(x, t)}{(p-2)t}.$$

Using, that u and \tilde{u} differ by a constant, and (13) in (12) we can conclude that

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial \tilde{u}(x, t)}{\partial t} \geq -\frac{\tilde{u}(x, t)}{(p-2)t}.$$

In particular, for $t \geq \frac{1}{2}$, it follows that

$$\frac{\partial u(x, t)}{\partial t} \geq -\frac{2}{p-2} \inf_{Q_1^+} \tilde{u}(x, t).$$

Remembering that $\tilde{u} = u + M = u + (CN + \|\phi\|_{L^\infty})$ we can conclude that

$$\frac{\partial u(x, t)}{\partial t} \geq -C(p)(N + \|\phi\|_{L^\infty} + \|f\|_{L^\infty}).$$

□

The corollary below is immediate.

Corollary 9. *Let $p \in (2, \infty)$. Assume the hypotheses of Lemma 8 and that $\phi \in C^2(Q_1^-)$. Then for any point $(y, s) \in \Gamma \cap Q_{1/2}^-$ and for $r < 1/4$*

$$\sup_{(x,t) \in Q_r^-(y,s)} |u(x, t) - u(y, s) - (x - y) \cdot \nabla u(y, s)| \leq Cr^q, \quad q = \frac{p}{p-1}.$$

where $C = C(p, \|u\|_{L^\infty(Q_1^-)}, \inf_{Q_1^-} f_t, \|\phi\|_{C^2(Q_1^-)})$.

5. APPENDIX

In this section, we recall some well known facts. The proposition below states that if the obstacle is in $C^{1,\beta}(B_1)$ then any bounded solution to the p -obstacle problem is locally in $C^{1,\alpha}$ for some α , see [26] and [21].

Proposition 10. *Let $p \in (1, \infty)$ and u solve the p -obstacle problem in B_1 with $\phi \in C^{1,\beta}(B_1)$ as obstacle. Then there is $\alpha(p, \|u\|_{L^\infty(B_1)}, \|\phi\|_{C^{1,\beta}(B_1)})$ such that*

$$\|u\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \leq C(p, \|u\|_{L^\infty(B_1)}, \|\phi\|_{C^{1,\beta}(B_1)}).$$

It is also well known that the solution to the obstacle problem for a uniformly elliptic operator with C^α coefficients leaves the obstacle in a $r^{1+\gamma}$ -fashion if the obstacle is $C^{1,\gamma}$ -regular. This follows for instance from Corollary 2.6 in [7].

Proposition 11. *Let u be a solution of the following obstacle problem: The smallest u such that*

$$\begin{cases} \operatorname{div}(\mathbf{A}\nabla u) \leq 0 \\ u \geq \phi \end{cases} \quad \text{in } B_1$$

where $\phi \in C^{1,\gamma}(B_1)$ with $\gamma \in (0, 1]$, $\mathbf{A} \in C^\alpha(B_1)$ and

$$\lambda|\xi|^2 \leq \mathbf{A}\xi \cdot \xi \leq \Lambda|\xi|^2, \quad 0 < \lambda < \Lambda$$

for all $\xi \in \mathbb{R}^n$. Then for $y \in \partial\{u > \phi\}$ and $r < 1/2$

$$\sup_{B_r(y)} |u(x) - u(y) - (x - y) \cdot \nabla \phi(y)| \leq Cr^{1+\gamma},$$

where $C = C(p, \|u\|_{L^\infty(B_1)}, \|\phi\|_{C^{1,\gamma}(B_1)}, \|\mathbf{A}\|_{C^\alpha(B_1)}, \Lambda, \lambda)$.

In the parabolic setting we have similar results. They follow below.

Proposition 12. *Let $p \in (1, \infty)$ and u solve the p -parabolic obstacle problem in Q_1^- with $\phi \in C^2(Q_1^-)$ as obstacle. Then there are constants $\alpha(p, \|u\|_{L^\infty(Q_1^-)}, \|\phi\|_{C^2(Q_1^-)})$ and $C(p, \|\phi\|_{C^2(Q_1^-)})$ such that*

$$\|u\|_{C_x^{1,\alpha}(Q_{\frac{1}{2}}^-)} + \|u\|_{C^\alpha(Q_{\frac{1}{2}}^-)} \leq C\|u\|_{L^\infty(Q_1^-)}.$$

The proposition above can be found in [9]. As a corollary we have the following scaled time estimate for solutions which will be useful in what follows, cf. [16]. We recall the notation

$$Q_r^-(x, t) = B_r(x, t) \times (-r^q + t, t], \quad q = \frac{p}{p-1}.$$

Corollary 13. *Let $p \in (2n/(n+2), \infty)$ and u solve the p -parabolic obstacle problem in Q_R^- with $\phi \in C^2(Q_1^-)$ as obstacle. Then there are constants $\alpha(p)$ and $C(p, \|\phi\|_{C^2(Q_R^-)})$ such that*

$$\|u(x, \cdot)\|_{C^\alpha((-R/2)^q, 0]} \leq \frac{C}{R^{\alpha q}} \|u\|_{L^\infty(Q_R^-)}.$$

Below we state a well known result for uniformly parabolic operators with Hölder continuous coefficients. In [5], the result is proved in the one-dimensional case for an operator of the type

$$a(x, t) \frac{\partial u}{\partial x^2} + b(x, t) \frac{\partial u}{\partial x} + c(x, t)u - \frac{\partial u}{\partial t},$$

with a , b and c uniformly Hölder continuous, see Theorem 2.1. However, the proof, which relies upon Lemma 2.2 in the very same paper, works perfectly fine, line by line, for the n -dimensional case and when the operator is as below.

Proposition 14. *Let u be a solution of the following obstacle problem*

$$\begin{cases} \max(\operatorname{div}(\mathbf{A}(x, t)\nabla u) - u_t, u - \phi) = 0 & \text{in } Q_1^- \\ u = g & \text{on } \partial_p Q_1^- \end{cases}$$

where $\phi \in C^2(Q_1^-)$, $\mathbf{A} \in C^\alpha(Q_1^-)$ for some $\alpha \in (0, 1)$ and

$$\lambda|\xi|^2 \leq \mathbf{A}(x, t)\xi \cdot \xi \leq \Lambda|\xi|^2, \quad 0 < \lambda < \Lambda,$$

for all $\xi \in \mathbb{R}^n$ and all $(x, t) \in Q_1^-$. Then for $(y, s) \in \partial\{u > \phi\}$ and $r < 1/2$

$$\sup_{Q_r^-(y, s)} |u(x, t) - u(y, s) - (x - y) \cdot \nabla \phi(y, s)| \leq Cr^q,$$

where $q = \frac{p}{p-1}$ and $C = C(p, \|u\|_{L^\infty(B_1)}, \|\phi\|_{C^2(Q_1^-)}, \|\mathbf{A}\|_{C^\alpha(Q_1^-)}, \Lambda, \lambda)$.

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